

# Calculation of the regularized vacuum energy in cavity field theories

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**Abstract.** A novel technique based on Schwinger's proper time method is applied to the Casimir problem of the M.I.T. bag model. Calculations of the regularized vacuum energies of massless scalar and Dirac spinor fields confined to a static and spherical cavity are presented in a consistent manner. While our results agree partly with previous calculations based on asymptotic methods, the main advantage of our technique is that the numerical errors are under control. Interpreting the bag constant as a vacuum expectation value, we investigate potential cancellations of boundary divergences between the canonical energy and its bag constant counterpart in the fermionic case. It is found that such cancellations do not occur.

## 1 Introduction

The effects of a classical static background field on the observables of a relativistic quantum field theory have been investigated in detail for quite some time. Perhaps the easiest way of distorting a free quantum field is to impose boundary conditions on a static surface. For the special case of an electromagnetic field, obeying boundary conditions on two parallel, uncharged and static plates, an attractive force was derived by Casimir as early as 1948 [1]. About two decades later, great efforts were made to investigate more complicated geometrical arrangements of the boundaries for a variety of free field theories and boundary conditions [2]. In particular, following the development of the bag models of hadrons [3–5], there has been increased interest in vacuum energies arising as a consequence of the boundary conditions on a static sphere. More recently, such calculations have been extended to include interacting, renormalizable quantum field theories in the framework of perturbation theory [6].

There are a number of approaches to calculating vacuum expectation values of field operators in relativistic quantum field theories. In general, one can classify these into two categories: local or global methods [2]. The global mode summation techniques avoid the explicit occurrence of divergences by analytical continuation (Zeta function regularization [7], heat kernel expansion [8]), whereas local methods take advantage of the fact that residual local

divergences<sup>1</sup> manifest themselves only as nonintegrable infinities of density functions on the boundary [9].

The main purpose of this paper is to present a consistent calculation of the regularized vacuum energy of massless scalar and Dirac spinor fields, that are confined to a static and spherical cavity, using the local stress tensor method [9–11]. The coefficients of the leading boundary divergences of the cavity energy have already been found elsewhere [10,12,13] based on asymptotic expansions. However, as one of these authors points out [13], the validity of this method can be questioned. It is therefore useful to check these results using an alternative numerical procedure. Moreover, the method developed below gives also insights into the order of magnitude of the coefficients of subleading divergences and finite parts. In principle, these can be calculated to any given accuracy, but in practice constraints on the computational effort can cause the errors to remain quite large.

In the framework of the M.I.T. bag model [3], there is for each field (scalar, fermionic and vector) a linear boundary condition that determines the eigenmodes and their energies. There is also a nonlinear boundary condition that guarantees the balance of the pressures at the boundary. The latter is usually taken into account in an average fashion, by minimizing the total energy of the bag with respect to the bag radius. Comparing the result of this minimization with the experimental data, a phenomenological value for the bag constant is obtained. However, the nonlinear boundary conditions of the model can also be used to interpret the bag constant as a vacuum

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<sup>1</sup> The free-space global divergence which renormalizes the cosmological constant is, in this case, already subtracted. This corresponds to the condition that the vacuum expectation values vanish in the case of a free-space theory.

expectation value which makes this additional and rather artificial minimization condition redundant [12].

This paper is organized as follows: In Sect. 2, we illustrate the local stress tensor method in the case of the straight-forward example of scalar and Dirac spinor fields satisfying linear boundary conditions on a plane. Well-known results are reproduced using the momentum-space representations of the free-space propagators, the reflection method [14], and Schwinger's spectral  $z$ -forms [15–22]. Section 3 contains a derivation of the expressions for the 00-component of the canonical energy-momentum tensor, integrated over the angles, for Klein-Gordon and Dirac fields satisfying the linear boundary conditions of the M.I.T. bag model on a static, spherical surface. Using the cavity mode representation of the propagators [15, 23, 24], the numerical evaluation is done by summing over the cavity mode quantum numbers first and integrating the resulting spectral  $z$ -form subsequently. An integration over a regularization volume yields the canonical regularized energy. Thereby an expansion in terms of the regularization parameter is derived from the expansion of the density into a Laurent type series [9]. In Sect. 4, the expression for the fermionic bag constant is derived and evaluated in analogy to Sect. 3. In the last section, we summarize our results and compare them with those available in the literature, and subsequently, we discuss problems and speculate on their possible solutions.

## 2 Vacuum energy in half-space field theories

In this section a method for calculating the vacuum expectation value of the canonical energy density  $\theta^{00}$  is illustrated for half-space field theories. Let us consider a massless scalar field subject to Dirichlet (D) and Neumann (N) boundary conditions, and a massless Dirac field satisfying the linear boundary condition of the M.I.T. bag model (MIT,q) [3]

$$\text{D: } \phi(x)|_{x \in S} = 0 \quad (1)$$

$$\text{N: } \partial_\mu n^\mu \phi(x)|_{x \in S} = 0 \quad (2)$$

$$\text{MIT, q: } (i n_\mu \gamma^\mu - 1) \psi(x)|_{x \in S} = 0, \quad (3)$$

$$n := (0, 0, 0, 1), \quad S := \{x | x^3 = 0\}.$$

Using the reflection method [14], the half-space propagators read

$$\Delta_{\text{D}}(x, y) = \Delta^0(x - y) - \Delta^0(x - y^T) \quad (4)$$

$$\Delta_{\text{N}}(x, y) = \Delta^0(x - y) + \Delta^0(x - y^T) \quad (5)$$

$$S_{\text{MIT, q}}(x, y) = S^0(x - y) - i\gamma^3 S^0(x - y^T), \quad (6)$$

$$y^T = (y^0, y^1, y^2, -y^3),$$

where  $S^0$  and  $\Delta^0$  stand for the free-space propagators. The structure of the canonical energy-momentum tensor  $\theta^{\mu\nu}$  (see for example [25]) requires that, in order to obtain the (diverging) vacuum energy densities, some bilinear operators must be applied to the corresponding propagators. For a massless Klein-Gordon field, we obtain

$$\langle \theta^{00}(x) \rangle = \lim_{y \rightarrow x} \partial_{x^0} \partial_{y^0} \Delta(x, y), \quad (7)$$

whereas, in the case of a massless Dirac field, the canonical vacuum energy density reads

$$\langle \theta^{00}(x) \rangle = \lim_{y \rightarrow x} \frac{i}{2} \text{Tr} [(\partial_{x^0} - \partial_{y^0}) \gamma^0] S(x, y), \quad (8)$$

where  $\Delta(x, y)$  and  $S(x, y)$  denote the field propagators, respectively.

Let us now give a general illustration of the  $z$ -form method for the calculation of the vacuum expectation value of local field bilinears. As an example, we concentrate on the expectation value of a Lorentz scalar in a scalar field theory, characterized by the bilinear  $\mathcal{B}$ .<sup>2</sup> This scalar quantity can be expressed in terms of the identity

$$\langle \mathcal{B}(\phi(x), \phi(x)) \rangle = \lim_{y \rightarrow x} \mathcal{B}_{x,y} \Delta^0(x - y). \quad (9)$$

Here  $\Delta^0(x - y)$  stands for the free-space Feynman propagator, and  $\mathcal{B}_{x,y}$  denotes the point splitted version of  $\mathcal{B}$ .<sup>3</sup> In momentum space, the propagator is given by

$$\Delta^0(p) \propto \frac{1}{p^2}, \quad (10)$$

and the application of  $\mathcal{B}$  results in

$$\mathcal{B}_{x,y} \frac{e^{-ip(x-y)}}{p^2} \propto \frac{b(p^2)}{p^2} e^{-ip(x-y)}, \quad (11)$$

where  $b$  is a polynomial in  $p^2$  (in a scale invariant theory  $b$  is just a power in  $p^2$  corresponding to the mass dimension of  $\mathcal{B}(\phi(x), \phi(x))$ ). We now take the limit  $y \rightarrow x$  and rotate to Euclidean momentum space

$$p_0 \rightarrow ip_0.$$

The (Euclidean) denominator can then be elevated into an exponent

$$\frac{1}{p^2} \rightarrow \int_0^\infty dz e^{-zp^2}, \quad (12)$$

and the remaining task is to integrate over the (Euclidean) momentum which involves (modified) Gaussian integrals

$$\int \frac{dp_\mu}{(2\pi)} \{1, (p_\mu)^2, \dots\} e^{-z(p_\mu)^2} = \left\{1, \frac{1}{2z}, \dots\right\} \frac{1}{\sqrt{4\pi z}}, \quad \mu = 0, \dots, 3. \quad (13)$$

Calculating  $\langle \theta^{00} \rangle$  for a massless scalar field, distorted by a Dirichlet plate at  $x^3 = 0$ , is now straightforward. The full propagator for this problem reads

$$\Delta(x, y) = \Delta^0(x - y) - \Delta^0(x - y^T), \quad (14)$$

$$y^T = (y^0, y^1, y^2, -y^3).$$

<sup>2</sup> The Dirac case can be treated in straightforward analogy.

<sup>3</sup> A contribution of a product of field derivatives  $\partial_\mu \phi(x) \partial_\nu \phi(x)$  to  $\mathcal{B}$ , e.g., implies a contribution of  $\partial_{x^\mu} \partial_{y^\nu}$  to  $\mathcal{B}_{x,y}$ .

Renormalizing the cosmological constant, by applying the bilinear operator of (7) to the *difference* of the half- and free-space propagators, yields finite results except at the boundary. In the case of Dirichlet boundary conditions, we obtain

$$\begin{aligned} \langle \theta^{00}(x^3) \rangle_D &= -\frac{i}{(2\pi)^4} \int d^4p \frac{(p^0)^2}{p^2 + i0} e^{-2ix^3p^3} \\ &= \frac{1}{(2\pi)^4} \int_e d^4p \frac{(p^0)^2}{p^2} e^{-2ix^3p^3} \\ &= \frac{1}{32\pi^2} \int_0^\infty dz e^{-(x^3)^2/z} z^{-3} \\ &= \frac{1}{32\pi^2 (x^3)^4}, \end{aligned} \quad (15)$$

while for Neumann boundary conditions we have

$$\langle \theta^{00}(x^3) \rangle_N = -\frac{1}{32\pi^2 (x^3)^4}. \quad (16)$$

In the case of a massless Dirac field, subject to the linear boundary condition of the M.I.T. bag model for the quark field (see (1)–(3), we obtain

$$\langle \theta^{00}(x^3) \rangle_{\text{MIT},q} \equiv 0. \quad (17)$$

This result is a consequence of the tracelessness of the energy-momentum tensor for this field [9].

### 3 Vacuum energy in cavity field theories

In this section, we shall calculate the canonical energy density of the vacua of massless scalar and Dirac fields, confined to a static and spherical cavity with radius  $R$ . After a brief description of the methods, the regularized vacuum energy is calculated for each field.

#### 3.1 Massless scalar fields

In order to perform the free-space subtraction in the spherical symmetric case, we need to express the free scalar propagator in terms of angular momentum eigenstates using the Rayleigh expansion for plane waves

$$\begin{aligned} \Delta^0(x-y) &= -\frac{i}{(2\pi)^4} \int d^4p \frac{e^{-ip(x-y)}}{p^2 + i0} \\ &= -(4\pi)^2 \frac{i}{(2\pi)^4} \int d^4p \frac{1}{p^2 + i0} \\ &\quad \times \sum_{\substack{l,l' \\ m,m'}} i^l (-i)^{l'} j_l(pr) j_{l'}(pr') Y_{l,m}(\hat{r}) \\ &\quad \times Y_{l,m}^*(\hat{p}) Y_{l',m'}^*(\hat{r}') Y_{l',m'}(\hat{p}). \end{aligned} \quad (18)$$

Applying the bilinear operator of (7) to this representation of the propagator, and performing in turn the angular

integration, the summation over  $m$  and a Euclidean rotation yields, after introducing the  $z$ -integration according to (12),

$$\begin{aligned} \langle \theta^{00}(r) \rangle &= \frac{1}{2\pi^{5/2}} \int_0^\infty dz \int_0^\infty dk k^2 \sum_l (2l+1) \\ &\quad \times (j_l(kr))^2 \frac{e^{-zk^2}}{z^{3/2}}. \end{aligned} \quad (19)$$

For the cavity part, we obtain the vacuum expectation value of the canonical energy density by applying the bilinear operator of (7) to the cavity mode representation of the propagator (see Appendix A.1). A  $z$ -integration is introduced, which originates from a shift of the (Euclidean) momentum squared denominator of the propagator into an exponential, as discussed in the preceding section. Here the meaning of the term momentum differs somewhat from that of free space due to the boundary conditions, i.e. the analogue to the expression

$$p^2 = (p^0)^2 - (\mathbf{p})^2 \quad (20)$$

in free space, is in the cavity

$$(p_{n,l})^2 = \omega^2 - (\varepsilon_{n,l})^2, \quad (21)$$

where  $\omega$  denotes the arbitrary (off-shell) energy, and  $\varepsilon_{n,l}$  stands for the energy of the mode, labelled by the radial quantum number  $n$  and angular momentum quantum number  $l$ .

The result of the free-space subtracted canonical vacuum energy density in the cavity is then

$$\begin{aligned} \langle \tilde{\theta}^{00}(r) \rangle_{\text{D,N}} &:= 4\pi \langle \theta^{00}(r) \rangle_{\text{D,N}} = \\ &= -\frac{1}{4\pi^{1/2}} \int_0^\infty dz \frac{1}{z^{3/2}} \sum_l \\ &\quad \times \left\{ \sum_{n,\mu} \int d\Omega [a_{n,l,\mu}^{D,N}(\mathbf{x}) a_{n,l,\mu}^{*D,N}(\mathbf{x})] e^{-z\varepsilon_{n,l}^2} \right. \\ &\quad \left. - \frac{2(2l+1)}{\pi} \int_0^\infty dk k^2 (j_l(kr))^2 e^{-zk^2} \right\} \\ &= -\frac{1}{4\pi^{1/2}} \int_0^\infty dz \frac{1}{z^{3/2}} \sum_l (2l+1) \\ &\quad \times \left\{ \sum_n \frac{1}{R^3} \mathcal{N}_{n,l}^{D,N} (j_l(|\varepsilon_{n,l}|r))^2 e^{-z\varepsilon_{n,l}^2} \right. \\ &\quad \left. - \frac{2}{\pi} \int_0^\infty dk k^2 (j_l(kr))^2 e^{-zk^2} \right\}. \end{aligned} \quad (22)$$

Here the  $a_{n,l,\mu}^{D,N}(\mathbf{x})$  denote the scalar cavity modes for either Dirichlet or Neumann boundary conditions, and the  $\mathcal{N}_{n,l}^{D,N}$  stand for their normalization constants [15,24], respectively, as explained in Appendix A.1. Using the plane-wave representation of the free-space propagator implies a  $z^{-3}$  divergence in the free-space part of (22). Since finite sums over linearly independent functions cannot change

the divergence structure common to all of these terms, a numerical evaluation of  $\langle \tilde{\theta}^{00}(r) \rangle_{D,N}$  would diverge.

The calculation reveals a substantial difference between the Dirichlet and the Neumann case. The Dirichlet boundary condition yields a  $z$ -form that is integrable at  $z = 0$ , whereas the Neumann boundary condition leads to a non-integrable  $z^{-3/2}$  divergence. We are able to show that this divergence is a global one, i.e. independent on  $r' := r/R$ . It therefore resembles another volume divergence (as does the free-space global divergence), and hence it can be omitted through a renormalization of the cosmological constant.

### 3.2 Massless Dirac fields

Using (8), the mode summation representing the cavity Dirac propagator [15,24], the spherical representation of the free-space propagator, and introducing the Schwinger  $z$ -integral, we obtain, after an angular integration,

$$\begin{aligned} \langle \tilde{\theta}^{00}(r) \rangle_{\text{MIT,q}} &:= 4\pi \langle \theta^{00}(r) \rangle_{\text{MIT,q}} \\ &= \frac{1}{2} \frac{1}{\pi^{1/2}} \int_0^\infty dz \frac{1}{z^{3/2}} \\ &\quad \left\{ \sum_\kappa \frac{1}{2} \sum_{n,\mu} \int d\Omega [q_{n,\kappa,\mu}^\dagger(\mathbf{x}) q(\mathbf{x})_{n,\kappa,\mu}] e^{-z\varepsilon_{n,\kappa}^2} \right. \\ &\quad \left. - \sum_l \frac{4}{\pi} (2l+1) \int_0^\infty dk k^2 (j_l(kr))^2 e^{-zk^2} \right\} \\ &= \frac{1}{2} \frac{1}{\pi^{1/2}} \int_0^\infty dz \frac{1}{z^{3/2}} \\ &\quad \left\{ \sum_\kappa \frac{1}{2} \sum_n \frac{1}{R^3} \mathcal{N}_{n,\kappa}^2 (2J+1) ((j_l(|\varepsilon_{n,\kappa}|r))^2 \right. \\ &\quad \left. + (j_{\bar{l}}(|\varepsilon_{n,\kappa}|r))^2) e^{-z\varepsilon_{n,\kappa}^2} \right. \\ &\quad \left. - \sum_l \frac{4}{\pi} (2l+1) \int_0^\infty dk k^2 (j_l(kr))^2 e^{-zk^2} \right\}, \quad (23) \end{aligned}$$

where

$$J := |\kappa| - \frac{1}{2}, \quad (24)$$

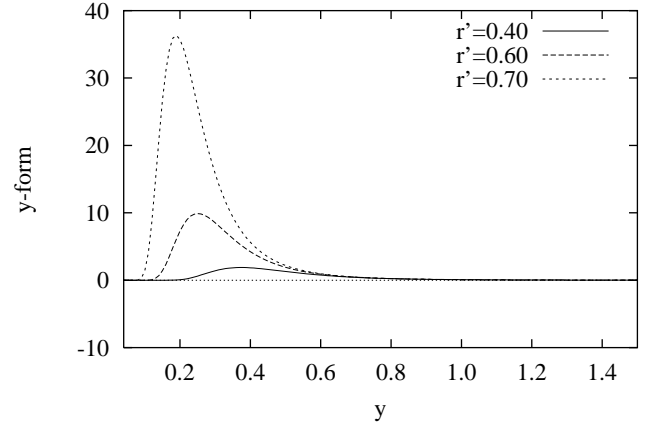
$$l := |J| + \frac{1}{2} \text{sgn } \kappa, \quad (25)$$

$$\bar{l} = l - \text{sgn } \kappa. \quad (26)$$

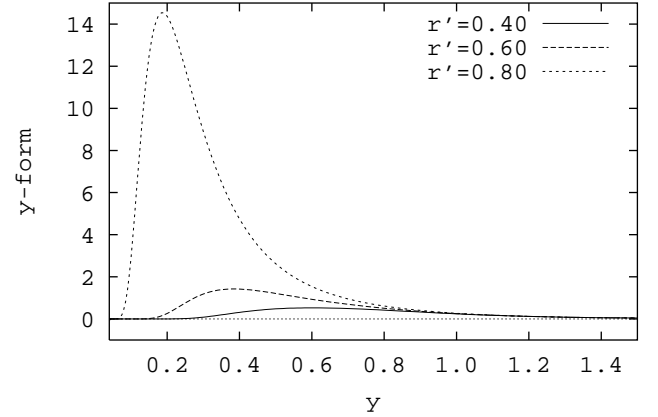
Here,  $q_{n,\kappa,\mu}(\mathbf{x})$  denotes the Dirac cavity mode, labelled with the radial quantum number  $n$ , the Dirac quantum number  $\kappa$ , and the angular momentum projection  $\mu$ , as discussed in Appendix A.2.

### 3.3 Numerical evaluation of $\langle \tilde{\theta}^{00} \rangle$

Equations (22) and (23) are suitable for a numerical evaluation. The  $k$  integration is performed in the range from



**Fig. 1.**  $y$ -forms for  $\langle \theta^{00} \rangle$  at three different locations  $r' = r/R$  for a massless scalar field fulfilling Dirichlet boundary conditions on a static sphere with radius  $R$

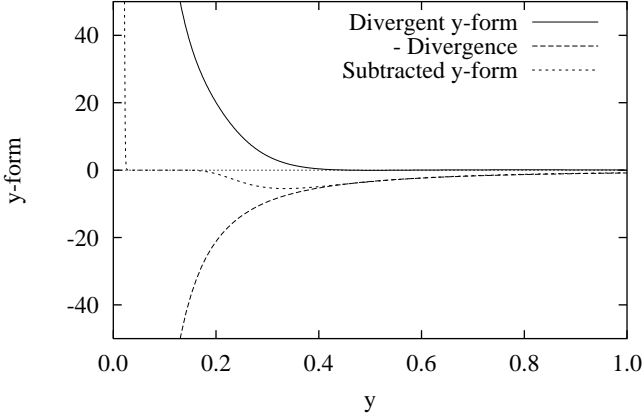


**Fig. 2.** Massless fermion field  $y$ -forms at three different locations  $r' = r/R$ . The field modes fulfill the linear boundary condition of the M.I.T. bag model on a static sphere with radius  $R$

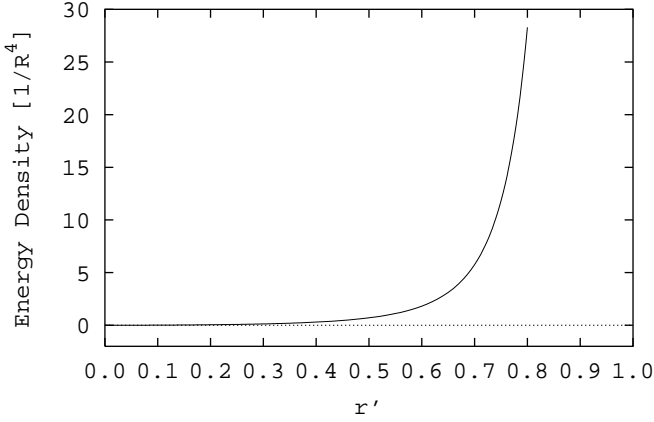
zero to  $\varepsilon_{\text{max}}$ , the maximal energy eigenvalue used in the sum over cavity modes. In our computations it is typically of the size  $200/R$ .

The calculation of  $\langle \tilde{\theta}^{00} \rangle$  is done in two steps. At first, we compute the  $z$ -form for the corresponding field and linear boundary condition at a number  $M$  of points  $r'$  ( $M \approx 500$  and  $r' := r/R$ ). It thereby proves convenient to make the variable substitution  $z = y^2$  resulting in a pure  $y^{-2}$  divergence in the  $y$ -form. In a second step, we integrate the regular part of the  $y$ -form and hence determine the energy density as a function of the position  $r'$ .

In Figs. 1 and 2, the  $y$ -forms for  $\langle \tilde{\theta}^{00} \rangle$  are displayed for the scalar Dirichlet and the Dirac case, respectively. It is shown numerically that the region of small  $y$ , where the form is practically zero, moves to the left with increasing maximal energy  $\varepsilon_{\text{max}}$ , whereas points to the right of this region remain unchanged. The  $y$ -form for  $\langle \tilde{\theta}^{00} \rangle$  in the scalar Neumann case contains a (under an increase



**Fig. 3.** Unsubtracted and subtracted  $y$ -forms for  $\langle\theta^{00}\rangle$  at  $r' = 0.5$  for a massless scalar field fulfilling Neumann boundary conditions on a static sphere with radius  $R$ . The subtracted divergence is of the form  $m/y^2$ , where  $m = 0.8463$



**Fig. 4.** Vacuum energy density for a massless scalar field fulfilling Dirichlet boundary conditions on a static sphere with radius  $R$

of  $\varepsilon_{\max}$ ) stable and global, i.e. independent of  $r'$ ,  $y^{-2}$ -divergence (see Fig. 3). The subtraction of this divergence amounts to renormalizing the cosmological constant and results in a  $y$ -form shown in Fig. 3.

The error  $f(r', \varepsilon_{\max})$  of  $\langle\tilde{\theta}^{00}\rangle$  due to the truncation of the sum over cavity energies in (22) and (23) can be determined by varying the cut-off energy  $\varepsilon_{\max}$ . We can safely estimate the upper bound for  $f(r', \varepsilon_{\max})$  at  $10^{-6}$ , valid for all  $r'$  and all fields and boundary conditions under consideration. As an example, Fig. 4 shows  $\langle\tilde{\theta}^{00}(r')\rangle$  for a Dirichlet scalar field.

Following Deutsch and Candelas [9], we may expand  $\langle\tilde{\theta}^{00}\rangle$  into a Laurent type series around  $r' = 1$

$$\langle\tilde{\theta}^{00}\rangle(\delta') = \frac{1}{R^4} \left[ c_{-4} (\delta')^{-4} + c_{-3} (\delta')^{-3} + \dots + c_{-1} (\delta')^{-1} + c_0 + \dots + c_N (r')^N \right], \quad (27)$$

$$N > 0, \quad \delta' := 1 - r'.$$

To extract the coefficients of the negative powers in (27), the function

$$E^L(\delta') := c_{-4} (\delta')^{-4} + \dots + c_{-1} (\delta')^{-1}$$

is fitted to the calculated curve in the interval

$$\delta'_{\min} \leq \delta' \leq \delta'_{\max},$$

where  $\delta'_{\min}$  and  $\delta'_{\max}$  are close to zero. For the determination of the positive power coefficients in (27), we fit the polynomial

$$E^T(r') := a_0 + a_1 r' + \dots + (r')^N$$

to the calculated curve within the interval

$$0 \leq r' \leq r'_{\max} := 1 - \delta'_{\max}.$$

A value of  $N = 9$  is used to obtain negligible fitting errors. Fitting  $E^L(\delta'(r'))$  to the polynomial

$$b_0 + b_1 r' + \dots + b_N (r')^N$$

within the above interval, the expression for  $\langle\tilde{\theta}^{00}\rangle$  reads

$$\begin{aligned} \langle\tilde{\theta}^{00}(r')\rangle &= E^L(\delta'(r')) + \Delta E(r'), \\ \Delta E(r') &:= c_0 + c_1 r' + \dots + c_N (r')^N, \\ c_i &:= a_i - b_i, \quad (0 \leq i \leq N). \end{aligned} \quad (28)$$

The numerical errors of the coefficients  $c_{-4}, \dots, c_{-1}$  can be estimated by varying the interval

$$\delta'_{\min} \leq \delta' \leq \delta'_{\max}.$$

In our computations we vary  $\delta'_{\min}$  from 0.05 to 0.15 and  $\delta'_{\max}$  from 0.1 to 0.2 to obtain errors of about 1%, 10% and up to 130% for the coefficients of the leading, next to leading, and the weakest divergences, respectively.

In the literature, there have been several attempts to extract boundary divergences of  $\langle\tilde{\theta}^{00}\rangle$  or the canonical energy  $E(\varepsilon')$  using the asymptotic properties of analytic functions [9, 10, 12, 26]. The drawback of these expansions lies in the fact that one does not know the errors of the analysis [13]. In our calculation of the regularized energy, we start with a determination of  $\langle\tilde{\theta}^{00}\rangle$ , where the only source of substantial errors is the fitting procedure to the Laurent type series. However, these errors can be estimated and made smaller in more extensive numerical calculations.

### 3.4 The regularized canonical vacuum energy

Following Bender and Hays [10], we regularize the canonical energy  $E$  by integrating  $\langle\tilde{\theta}^{00}(r')\rangle$  only to an upper limit of

$$r'_{\max} = (1 - \varepsilon'), \quad \varepsilon' := \frac{\varepsilon}{R},$$

where  $\varepsilon$  denotes the distance from the boundary. Using (27), we obtain

$$\begin{aligned} E(\varepsilon') &= \frac{1}{R} \int_0^{1-\varepsilon'} dr' r'^2 \langle \tilde{\theta}^{00}(r') \rangle \\ &= \frac{1}{R} \left[ \tilde{c}_{-3} \varepsilon'^{-3} + \tilde{c}_{-2} \varepsilon'^{-2} + \tilde{c}_{-1} \varepsilon'^{-1} \right. \\ &\quad \left. + \tilde{c}_{\log} \log \varepsilon' + \tilde{c}_0 + \mathcal{O}(\varepsilon') \right]. \end{aligned} \quad (29)$$

The coefficients  $\{c\}$  of (27) and the coefficients  $\{\tilde{c}\}$  of (29) are related by

$$\begin{aligned} \tilde{c}_{-3} &= \frac{1}{3} c_{-4}, \\ \tilde{c}_{-2} &= -c_{-4} + \frac{1}{2} c_{-3}, \\ \tilde{c}_{-1} &= c_{-4} - 2c_{-3} + c_{-2}, \\ \tilde{c}_{\log} &= -c_{-3} + 2c_{-2} - c_{-1}, \end{aligned} \quad (30)$$

for the divergent terms, whereas the coefficients of the positive powers of  $\varepsilon'$  depend also on the truncation number  $N$ . Since we are only interested in the limit  $\varepsilon' \rightarrow 0$ , the only substantial coefficient of  $\{\tilde{c}_j | j \geq 0\}$  is  $\tilde{c}_0$ , given by

$$\tilde{c}_0 = -\frac{1}{3} c_{-4} + \frac{3}{2} c_{-3} - \frac{3}{2} c_{-1} + \sum_{i=0}^N \frac{1}{i+3} c_i. \quad (31)$$

For the fermionic case, we may assume  $c_{-4} = 0$ . This is so for fermionic quantum fields because the canonical energy-momentum tensor coincides with the improved tensor, and hence it is traceless [9].

Table 1 contains a list of the coefficients  $\{\tilde{c}\}$  for the various fields and boundary conditions, where the errors are determined from the errors of the coefficients  $\{c\}$  according to the rules of error propagation. Note that the leading divergences for the scalar Dirichlet and Neumann fields exhibit a behavior analogous to that of the half-space problem, i.e., they are equal in magnitude and carry opposite signs.

## 4 The fermionic bag constant

The bag constant  $B$  is introduced into the Lagrangian to achieve conservation of a Poincaré generator, i.e. the energy [3]. In bag model calculations  $B$  is determined using the minimization condition

$$\frac{d}{dR} E(R) = 0,$$

where  $E(R)$  denotes the total energy of the bag. Milton [12] suggests that one should calculate the bag constant from first principles, by interpreting it as a vacuum expectation value. The nonlinear boundary condition of the model then serves as a definition for  $B$ . Here we will carry out the calculation of this vacuum expectation value for a massless fermion field, subject to the boundary conditions

of the M.I.T. bag model on a static sphere with radius  $R$ . The expression for  $B_{\text{MIT,q}}$  reads

$$B_{\text{MIT,q}} := -\frac{1}{2} \langle \partial_r (\bar{\psi}\psi) \rangle \Big|_{r=R} \quad (32)$$

and implies that the differential operator

$$-\lim_{y \rightarrow x} \frac{1}{2} \text{Tr} (\partial_{r_x} + \partial_{r_y}) \quad (33)$$

should be applied to the difference of the cavity and the free-space propagator. We expect  $B_{\text{MIT,q}}$  to be infinite. In order to regularize it, we compute it at some interior point a distance  $\varepsilon$  away from the boundary, rather than at some point on the boundary, as (32) demands. For the angular integrated version  $\tilde{B}_{\text{MIT,q}}(\varepsilon')$  of  $B_{\text{MIT,q}}(\varepsilon')$ , we arrive at

$$\begin{aligned} \tilde{B}_{\text{MIT,q}}(\varepsilon') &:= 4\pi B_{\text{MIT,q}}(\varepsilon') \\ &= -\frac{1}{\pi^{1/2}} \int_0^\infty dz \frac{1}{z^{1/2}} \sum_{\kappa} (2J+1) \sum_{n>0} \mathcal{N}_{n,\kappa}^2 \varepsilon_{n,\kappa}^2 e^{-z\varepsilon_{n,\kappa}^2} \\ &\quad \times \left\{ \frac{j_l(|\varepsilon_{n,\kappa}|Rr')}{2l+1} (l j_{l-1}(|\varepsilon_{n,\kappa}|Rr') \right. \\ &\quad - (l+1) j_{l+1}(|\varepsilon_{n,\kappa}|Rr')) \\ &\quad - \frac{j_{\bar{l}}(|\varepsilon_{n,\kappa}|Rr')}{2\bar{l}+1} (\bar{l} j_{\bar{l}-1}(|\varepsilon_{n,\kappa}|Rr') \\ &\quad \left. - (\bar{l}+1) j_{\bar{l}+1}(|\varepsilon_{n,\kappa}|Rr')) \right\}, \end{aligned} \quad (34)$$

with

$$\varepsilon' = \frac{\varepsilon}{R}, \quad r' := 1 - \varepsilon'.$$

There is no free-space part in (34) since  $\text{Tr} \gamma^\mu \equiv 0$  for  $\mu = 0, \dots, 3$ .

The result of the calculation of the coefficients  $\{\tilde{c}\}$  in the expansion of the regularized bag constant energy  $E_{\text{MIT,q}}^B(\varepsilon')$  with

$$\begin{aligned} E_{\text{MIT,q}}^B(\varepsilon') &:= \frac{R^3}{3} (1 - \varepsilon')^3 \times \tilde{B}_{\text{MIT,q}}(\varepsilon') \\ &= \frac{1}{R} \left[ \tilde{c}_{-4} \varepsilon'^{-4} + \tilde{c}_{-3} \varepsilon'^{-3} + \tilde{c}_{-2} \varepsilon'^{-2} \right. \\ &\quad \left. + \tilde{c}_{-1} \varepsilon'^{-1} + \tilde{c}_0 + \mathcal{O}(\varepsilon') \right] \end{aligned} \quad (35)$$

is displayed in Table 2. Here we also list the set of coefficients in the expansion of the total vacuum energy  $E_{\text{MIT,q}}^{\text{tot}}(\varepsilon')$  given by

$$\begin{aligned} E_{\text{MIT,q}}^{\text{tot}}(\varepsilon') &:= \int_0^{1-\varepsilon'} dr' (r')^2 \left\{ \langle \tilde{\theta}^{00}(r') \rangle_{\text{MIT,q}} - \tilde{B}_{\text{MIT,q}}(\varepsilon') \right\} \\ &= E_{\text{MIT,q}}(\varepsilon') - E_{\text{MIT,q}}^B(\varepsilon'). \end{aligned} \quad (36)$$

**Table 1.** The coefficients  $\{\tilde{c}\}$  for the divergent and the finite parts of the canonical vacuum energy  $E(\varepsilon')$  for massless scalar Dirichlet (D) and Neumann (N), and Dirac (MIT,q) fields

	$\tilde{c}_{-3}$	$\tilde{c}_{-2}$	$\tilde{c}_{-1}$	$\tilde{c}_{\log}$	$\tilde{c}_0$
D	$0.0132435 \pm 0.1\%$	$-0.0232 \pm 10\%$	$-0.0065 \pm 40\%$	$-0.0052 \pm 10\%$	$-10.2 \pm 50\%$
N	$-0.01318 \pm 0.5\%$	$-0.0012 \pm 60\%$	$-0.085 \pm 40\%$	$0.64 \pm 15\%$	$45 \pm 60\%$
MIT,q	0	$-0.01058 \pm 0.45\%$	$-0.35 \pm 100\%$	$-0.73 \pm 100\%$	$0.16 \pm 20\%$

**Table 2.** The coefficients  $\{\tilde{c}\}$  for the divergent and the finite part of  $E_{\text{MIT,q}}^B$  and the total vacuum energy  $E_{\text{MIT,q}}^{\text{tot}}(\varepsilon')$ 

	$\tilde{c}_{-4}$	$\tilde{c}_{-3}$	$\tilde{c}_{-2}$	$\tilde{c}_{-1}$	$\tilde{c}_{\log}$	$\tilde{c}_0$
$E_{\text{MIT,q}}^B$	$-0.1593 \pm 0.2\%$	$0.375 \pm 1\%$	$-0.119 \pm 5\%$	$0.11 \pm 20\%$	0	$-7.2 \times 10^5 \pm 50\%$
$E_{\text{MIT,q}}^{\text{tot}}$	$0.1593 \pm 0.2\%$	$-0.375 \pm 1\%$	$0.108 \pm 6\%$	$-0.46 \pm 85\%$	$-0.73 \pm 100\%$	$7.2 \times 10^5 \pm 50\%$

## 5 Summary and Discussion

The main purpose of this paper was to investigate the effect of a static and spherical boundary on the canonical vacuum energy densities of otherwise free massless Klein-Gordon and Dirac fields. Thereby the linear boundary conditions of the M.I.T. bag model [3] have been used. A Green's function method, that is based on the eigenmode representation of the propagator and the Schwinger parametrization of the Euclidean momentum squared denominator, has been used to obtain numerical results for the vacuum energy densities. For the fermionic field, a calculation of the regularized bag constant based on its definition via the nonlinear boundary condition was carried out. Numerical results for the densities have been fitted to Laurent type series in powers of the distance to the boundary. The expressions obtained in this manner could be integrated within a regularization volume to yield an expansion of the energies in terms of the regularization parameter. Numerical errors of the coefficients in these expansions turn out to be less than 1% for the leading singularity, and up to 100% for the weakest divergences. In general, the finite part  $\tilde{c}_0$  could be determined to about 50% accuracy, using this method and the stated computational effort.

We can compare our results directly with those in the literature. Bender and Hays [10] have calculated the leading divergence of the canonical part of the vacuum energy using a Green's function method. They introduced their regularization in the same fashion as we do, but they relied on analytical formulae for the asymptotic expansion of Bessel functions. For example in the fermionic and the Dirichlet scalar case, the comparison is as follows:

$$\begin{aligned}
 \text{Bender \& Hays : } \tilde{c}_{-2}^{\text{MIT,q}} &= -\frac{1}{120\pi} \approx -0.0027, \\
 \tilde{c}_{-3}^{\text{D}} &= -\frac{1}{24\pi} \approx -0.0133 \\
 \text{this work : } \tilde{c}_{-2}^{\text{MIT,q}} &= -0.01058 \pm 0.45\%, \\
 \tilde{c}_{-3}^{\text{D}} &= 0.01324 \pm 0.1\%
 \end{aligned} \tag{37}$$

There are, of course, disagreements, but the sign error of Bender and Hays' result for the scalar case has already indirectly been pointed out by Milton [13]. In his work

he mentions an overall sign error in the vector field mode sum of [10]. Since the mode sum of the transverse electric vector field is up to the  $l = 0$  contribution equal to that of the scalar Dirichlet case (compare (2.27) and (2.15) in [10]), an overall sign error implies a sign error in the result of the scalar Dirichlet case. The factor of four difference in the fermionic case might be due to an omission of the trace of  $\gamma_0^2$  as required by (8).

Olaussen and Ravndal [26] used a Green's function method to calculate the *electromagnetic* canonical vacuum energy density for a spherical bag, and we are thus not able to compare their results with ours.

There are two common types of regularization procedures. For example Milton [12, 13] calculated the canonical fermionic and vector field vacuum energy using a temporal regularization, whereas Bender and Hays [10] worked with the same volume regularization as we did. Temporal regularization derives from the idea of point splitting as a means of cutting off ultraviolet contributions to the energy density at each point in the cavity. On the other hand, the volume regularization used above avoids the integration of *boundary* divergences of the energy density. Therefore a strict comparison of results obtained in those two different regularization schemes should not be made. One would, however, expect the same divergence structure with different coefficients. An order of magnitude check of Milton's result in comparison with our result for  $\tilde{c}_{-2}$  in the fermionic case [12] reveals less than a factor of 10 difference. The large deviation in the finite parts (a factor of about  $10^2$ ) can very well be a consequence of the use of asymptotic expansions in [12]. In fact, as Milton himself points out in the conclusions of [13], even existing logarithmic divergences are then not recognized.

The small error of the leading infinite terms allows us to compare the results for the canonical and the bag constant part in the fermionic case. A cancellation of the infinities between these two contributions to the total energy does *not* occur, since there is a nonvanishing quartic divergence in the bag constant contribution, whereas the leading infinite term in the canonical part is quadratic (see Table 2). Moreover, the coefficient  $\tilde{c}_0$  in  $E_{\text{MIT,q}}^B$  is quite large.

One may speculate that, in a gauge theory calculated perturbatively, infinities arising from the canonical and the bag constant part of the energy should cancel each other, leaving a meaningful finite part. We plan to investigate this matter in the near future. If there is still no hint for a cancellation, one would argue that the introduction of a *constant* bag energy density  $B$  is a too naive device to achieve the conservation of the Poincaré generator, i.e. the energy. Perhaps, a locally conserved energy-momentum tensor (if at all definable on physical grounds) would reveal finite vacuum energies. On the other hand, using the canonical energy-momentum tensor together with soft boundaries, i.e. practically confining potentials (as for example a harmonic oscillator potential), could possibly bypass the global vacuum infinities. The drawbacks of this method are the enormous numerical effort and the necessity of a cutoff parameter to ensure confinement. Moreover, it is quite unsatisfactory, that this additional parameter would have to be determined experimentally. Along the same lines, there have been suggestions to introduce phenomenological parameters in the expression for the bag energy. These parameters could absorb the divergences. However, they should be determined experimentally [6, 12], which is again unsatisfactory.

A possible extension of the work done here would be a more extensive numerical calculation (determination of the energy density at points closer to the boundary) to achieve higher precision for the coefficients of the subleading divergences and the finite part.

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## A Cavity modes

Here, the eigenmodes and propagators for massless scalar and Dirac fields in a static spherical cavity of radius  $R$  are given [23, 24, 27].

### A.1 Massless scalar fields

The cavity modes of the massless scalar fields are given as

$$\phi_{n,l,\mu}(\mathbf{x}) = \frac{\mathcal{N}_{n,l}^{\text{D,N}}}{R^{3/2}} i j_l(\varepsilon_{n,l} R) Y_{l\mu}(\hat{r}) . \quad (38)$$

A mode representation of the propagator reads as

$$\begin{aligned} \Delta(x, x') &= i \sum_{n,l,\mu} \phi_{n,l,\mu}(\mathbf{x}) \phi_{n,l,\mu}^*(\mathbf{x}') \\ &\times \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(x_0 - x'_0)}}{\omega^2 - (\varepsilon_{n,l})^2 + i0} . \end{aligned} \quad (39)$$

The energy eigenvalues  $\varepsilon_{n,l}$  depend, of course, on the chosen boundary condition. In the Dirichlet case, we have the eigenvalue equation

$$j_l(\varepsilon_{n,l} R) = 0 , \quad (40)$$

and in the case of Neumann boundary conditions the eigenvalue equation is

$$l j_l(\varepsilon_{n,l} R) - \varepsilon_{n,l} R j_{l+1}(\varepsilon_{n,l} R) = 0 . \quad (41)$$

The normalization constants  $\mathcal{N}_{n,l}^{\text{D}}$  and  $\mathcal{N}_{n,l}^{\text{N}}$  are given as

$$\mathcal{N}_{m\Sigma}^{\text{D,N}} = \sqrt{\frac{2}{(\varepsilon_m^\Sigma R)^2 - J(J+1)}} \left| \frac{\varepsilon_m^\Sigma R}{j_J(\varepsilon_m^\Sigma R)} \right| , \quad (42)$$

$\Sigma = \mathcal{S}, \mathcal{L}, \mathcal{M}$

$$\mathcal{N}_{m\varepsilon}^{\text{D,N}} = \frac{\sqrt{2}}{|j_{J+1}(\varepsilon_m^\varepsilon R)|} . \quad (43)$$

### A.2 Massless fermion fields

The cavity modes for massless fermion fields satisfying the linear boundary condition of the M.I.T. bag model are given by the Dirac spinors

$$q_{n,\kappa,\mu}(\mathbf{x}) = \begin{pmatrix} g_{n,\kappa}(r) \chi_\kappa^\mu(\hat{r}) \\ i f_{n,\kappa}(r) \chi_{-\kappa}^\mu(\hat{r}) \end{pmatrix} , \quad (44)$$

where  $\chi_\kappa^\mu(\hat{r})$  is the usual two-component spherical spinor. Here  $n$ ,  $\kappa$ , and  $\mu$  denote the radial, Dirac, and magnetic quantum numbers respectively, and the radial functions  $g_{n,\kappa}(r)$  and  $f_{n,\kappa}(r)$  are given in terms of the spherical Bessel functions  $j_l$  by

$$g_{n,\kappa}(r) = \frac{\mathcal{N}_{n,\kappa}}{R^{3/2}} j_l(p_{n,\kappa} r) \quad (45)$$

$$f_{n,\kappa}(r) = \frac{\mathcal{N}_{n,\kappa} \operatorname{sgn}(n) \operatorname{sgn}(\kappa)}{R^{3/2}} j_{\bar{l}}(p_{n,\kappa} r) . \quad (46)$$

The discrete momenta  $p_{n,\kappa}$  in (45) and (46) are determined by the linear boundary condition

$$(i\boldsymbol{\gamma} \cdot \hat{r} + 1) q_{n,\kappa,\mu}(\mathbf{r})|_{r=R} = 0 \quad (47)$$

which leads to the eigenvalue equation

$$j_l(x_{n,\kappa}) + \operatorname{sgn}(n) \operatorname{sgn}(\kappa) j_{\bar{l}}(x_{n,\kappa}) = 0 . \quad (48)$$

The normalization constant  $\mathcal{N}_{n,\kappa}$  in (45) and (46) is given by

$$\mathcal{N}_{n,\kappa} = (2\omega_n(\omega_n + \kappa))^{-1/2} \left| \frac{x_n}{j_l(x_n)} \right| . \quad (49)$$

Here we have introduced the dimensionless energy and momentum parameters

$$x_{n,\kappa} = p_{n,\kappa} R, \quad (50)$$

$$\omega_{n,\kappa} = \operatorname{sgn}(n) x_{n,\kappa} . \quad (51)$$



The cavity propagator for massless fermions can be represented in terms of these cavity modes as

$$S(x, x') = i \sum_{\kappa\nu\mu} q_{n,\kappa,\mu}(\mathbf{x}) \bar{q}_{n,\kappa,\mu}(\mathbf{x}') \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(x_0-x'_0)}}{\omega - \varepsilon_n \pm i0}, \quad (52)$$

where the usual Feynman prescription for the poles is used.

## References

1. H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948)
2. G. Plunien, B. Müller, W. Greiner, Phys. Rep. **134**, 89 (1986)
3. A. Chodos et al., Phys. Rev. D **9**, 3471 (1974)
4. W. A. Bardeen et al., Phys. Rev. D **11**, 1094 (1975)
5. T. D. Lee, Phys. Rev. D **19**, 1802 (1979)
6. M. Bordag, D. Robaschik, E. Wieczorek, Ann. Phys. **165**, 192 (1985)
7. V. V. Nesterenko, I. G. Pirozhenko, Phys. Rev. D **57**, 1284 (1998)
8. M. Bordag, E. Elizalde, K. Kirsten, J. Math. Phys. **37**, 895 (1996)
9. D. Deutsch, P. Candelas, Phys. Rev. D **20**, 3063 (1979)
10. C. M. Bender, P. Hays, Phys. Rev. D **14**, 2622 (1976)
11. K. A. Milton, L. DeRaad, J. Schwinger, Ann. Phys. **115**, 388 (1978)
12. K. A. Milton, Phys. Rev. D **22**, 1444 (1980)
13. K. A. Milton, Phys. Rev. D **22**, 1441 (1980)
14. T. H. Hansson, R. L. Jaffe, Phys. Rev. D **28**, 882 (1983)
15. A. J. Stoddart, Ph.D. thesis, University of Cape Town, 1989
16. G. U. Schreiber, R. D. Viollier, Ann. Phys. **215**, 277 (1992)
17. G. U. Schreiber, R. D. Viollier, Phys. Lett. B **279**, 131 (1992)
18. A. J. Stoddart, R. D. Viollier, Nucl. Phys. A **532**, 657 (1991)
19. A. J. Stoddart, R. D. Viollier, Nucl. Phys. A **541**, 623 (1992)
20. P. R. Page, R. J. Lindebaum, R. D. Viollier, Nucl. Phys. A **506**, 1003 (1993)
21. J. A. Cuthbert, R. D. Viollier, Z. Phys. C **58**, 295 (1993)
22. R. D. Viollier, Prog. Part. Nucl. Phys. **36**, 251 (1996)
23. R. F. Buser, R. D. Viollier, P. Zimak, Int. J. Theor. Phys. **27**, 925 (1988)
24. M. S. O'Connor, R. D. Viollier, Ann. Phys. **248**, 268 (1996)
25. P. Pascual, R. Tarrach, **QCD: Renormalization for the Practitioner** (Springer Verlag, Heidelberg, 1984)
26. K. Olaussen, F. Ravndal, Nucl. Phys. B **192**, 237 (1981)
27. R. D. Viollier, S. A. Chin, A. K. Kerman, Nucl. Phys. A **407**, 269 (1983)